SOME COEFFICIENT ESTIMATES FOR H^p FUNCTIONS

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Abstract. We find the maximum modulus of the n-th Taylor coefficient c_n of a function in the unit ball of H^p, $1 \le p \le 1$, provided that c_0 is fixed, and identify the corresponding extremal functions.

1. Introduction

and c such that $0 < c < 1$: In the following sections, we consider only such values of p and c. In proving the main result, we prove some intermediate theorems which are of independent interest.

3. Statement of the Main Results

Theorem 3.1. *If $2^{1/p} \cdot c \le 1$; then*

$$M_p(n;c) = \frac{2}{p} c^{1-\frac{p}{2}} \sqrt{1-c^p}$$

and the corresponding extremal function is

$$f(z) = (c^{\frac{p}{2}} + \sqrt{1-c^p}\, z^n)^{\frac{2}{p}}:$$

Theorem 3.2. *If $0 < c < 2^{1/p}$; then the zero-free function f such that $\|f\|_p \le 1$ and $|f(0)| = c$ that maximizes $|f'(0)|$ is*

$$f(z) = 2^{1/p}(1+z)^{\frac{2}{p}}(2^{\frac1p}c)^{\frac{1-z}{1+z}}$$

and

$$f'(0) = c\left(\frac{2}{p} + \log \frac{1}{2^{\frac{2}{p}}c^2}\right):$$

Theorem 3.3. *If $0 < c < 2^{1/p}$; then*

$$M_p(n;c) = \left(\frac{2}{p} - 1\right)cv + \frac{c}{v}$$

and the corresponding extremal function is

$$f(z) = \frac{c}{v}(1 + vz^n)^{\frac{2}{p}-1}(v + z^n)$$

where v is the unique root ($0 < v \le 1$) of $v^p - c^p = c^p v^2$: In particular, for $p = 1$ and $0 < c < \frac12$; $M_1(n;c) = 1$ and $f(z) = c + z^n + cz^{2n}$:

4.   ):

Proposition 4.2. *The singular function $S(z)$ that maximizes $ReS^\theta(0)$ if $S(0) = c$ is*

$$S(z) = c^{1-}$$

We now have a variational problem, where we need to find

$$\max \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi'(t) \cos t\, dt$$

under the constraints $\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\psi(t)}\, dt = 1$ and $\frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(t)\, dt = \log c < 0.$

If this maximum equals λ and is attained when $\psi(t) = \psi_0(t)$, then ψ_0 also solves the following dual variational problem: find

$$\min \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\psi(t)}\, dt$$

under the constraints $\frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(t) \cos t\, dt = \lambda$ and $\frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(t)\, dt = \log c;$ because the above minimum is then equal to 1. To see this, suppose that for some $\psi(t) = \psi_1(t)$ satisying the constraints of the dual problem $\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\psi_1(t)}\, dt < 1.$ Then there is some $s > 0$ such that the function $\psi_2(t) = \psi_1(t) + s\cos t$ satisfies $\frac{1}{2\pi} \int_{-\pi}^{\pi} \psi_2(t) \cos t\, dt$ s >

Therefore,

$$\varphi'(v) \geq 0 \quad , \quad c^{i\frac{p}{2}} \mathcal{p} \frac{v^{p_i 1}}{v^p {}_i\, c^p} \geq 1 + \frac{1}{v^2}$$

$$, \quad (1 + v^2)^2 (c^p)^2 {}_i\ v^p (1 + v^2)^2 c^p + v^{2(p+1)} \geq 0$$

$$, \quad (c^p {}_i\ \frac{v^p}{1 + v^2})(c^p {}_i\ \frac{v^{2+p}}{1 + v^2}) \geq 0$$

When $c \geq 2^{i\frac{1}{p}}$, $\varphi'(v) \geq 0$ and the maximum of $\varphi(v)$ is obtained at $v = 1$:

$$\varphi(1) = \frac{2}{p} c^{1i\frac{p}{2}} \mathcal{p}\overline{1 {}_i\ c^p}.$$

In that case, the function

$$f(z) = (c^{\frac{p}{2}} + \mathcal{p}\overline{1 {}_i\ c^p} z)^{\frac{2}{p}}$$

is an element of H^p with norm 1 such that $f(0) = c$ and $f'(0) = \varphi(1)$.

When $n > 1$, use the function F described in Section 2. Since $f(z) = F(z^n)$, we obtain the extremal function

$$f(z) = (c^{\frac{p}{2}} + \mathcal{p}\overline{1 {}_i\ c^p} z^n)^{\frac{2}{p}}$$

with the same maximal n-th Taylor coefficient as in the case $n = 1$. ¤

Notice that f is a zero-free function, and therefore Theorem 3.1 also solves the extremal problem for zero-free H^p functions whose value at the origin is not too close to 0. Let us now consider zero-free functions in H^p whose value at the origin are small, as stated in Theorem 3.2.

Proof. Let $0 < c < 2^{i\frac{1}{p}}$ and let $f \in H^p$ be a non-zero function such that $f(0) = c$ and $\|f\|_p \cdot 1$ for which $|f'(0)|$ is maximal. Write $f(z) = S(z)F(z)$ where S is a singular function and F is an outer function. Writing $S(0) = u$ and $F(0) = v$, notice that by Proposition 4.3, $v \geq 2^{i\frac{1}{p}}$. Using the estimates given by Proposition 4.2 and Theorem 3.1, we get that

$$|f'(0)| \cdot v 2u \log\frac{1}{u} + u\frac{2}{p} v^{1i\frac{p}{2}} \mathcal{p}\overline{1 {}_i\ v^p}$$

$$= 2c\log\frac{v}{c} + \frac{2c}{p}\sqrt{\frac{1}{v^p} {}_i\ 1}$$

$$= \varphi(v)$$

One can easily show that $\varphi(v)$ is decreasing on $[2^{i\frac{1}{p}}, 1]$ and therefore attains its maximum at $v = 2^{i\frac{1}{p}}$. Therefore $u = c2^{\frac{1}{p}}$, and the function

$$f(z) = (2^{\frac{1}{p}}c)^{\frac{1-z}{1+z}} 2^{i\frac{1}{p}}(1 + z)^{\frac{2}{p}}$$

is a zero-free function such that $f(0) = c$, $\|f\|_p = 1$ and

$$f'(0) = \Phi'(2^{i\frac{1}{p}}) = c\left(\frac{2}{p} + \log\frac{1}{2^{\frac{2}{p}}c^2}\right).$$

□

We now consider functions in H^p that can have zeros and whose value at the origin is small, as stated in Theorem 3.3.

Proof. Consider the case $n = 1$ and let $f \in H^p$ be such that $\|f\|_p \leq 1$ and $f(0) = c$. Write $f(z) = B(z)F(z)$ where B is a Blaschke product with $B(0) = v > 0$, and F is zero-free with $F(0) = u$.

Suppose first that $u \geq 2^{i\frac{1}{p}}$. Then $c \leq v \leq 2^{\frac{1}{p}}c$, so by the proof of Theorem 3.1

$$|f'(0)| \leq c\left(\frac{1}{v} - v\right) + \frac{2}{p}c^{1-\frac{p}{2}}\frac{\Phi}{v^p - c^p}$$

$$= \Psi(v)$$

and

$$\Psi'(v) \geq 0 \iff \left(c^p - \frac{v^p}{1+v^2}\right)\left(c^p - \frac{v^{2+p}}{1+v^2}\right) \geq 0$$

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