JENSEN TYPE INEQUALITIES AND RADIAL NULL SETS

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Abstract. We extend Jensen's formula to obtain an upper estimate of $\log|f(0)|$ for analytic functions in the unit disk $\mathbb{D}$ that are subject to a growth restriction. Suppose we have a closed subset E of the unit circle and f in addition is continuous in the union of the open disk and E. We obtain a formula that gives an upper estimate of of $\log|f(0)|$ in terms of the values of f on E and the so-called k-entropy of E. When the set E is taken to be the whole unit circle, we get the classical Jensen's inequality. Our formula is then applied to the study of radial null sets. 2000 Mathematics Subject Classification: 30H05, 30E25, 46E15.

1 Growth Spaces

In what follows, k denotes an increasing twice differentiable function that maps $[0, 1)$ [illegible]

2 Two Problems

(A) Find good (upper and lower) estimates for the quantity

$$J(Z;k) = \sup\{\log|f(0)| : f \in UBA^{<k>}; f|_Z = 0\}$$

where $Z = \{a_n\} \subset \mathbb{D}$ is a given sequence.
(B) Find good estimates for

$$J(E;\varphi;k) = \sup\{\log|f(0)| : f \in UBA^{<k>} \cap C(\mathbb{D} \cup E); |f||_E = \varphi\}$$

where $E \subset \partial\mathbb{D}$ is a closed set and φ is a non-negative continuous function on $E.$
Note that for $k \equiv 0;$ $(A^{<0>} = H^\infty)$ both problems have exact solutions:

$$J(Z;0) = -\sum_n \log\frac{1}{|a_n|}$$

$$J(E;\varphi;0) = \int_E \log\varphi(\zeta)dm(\zeta)$$

where dm is the normalized Lebesgue measure on $\partial\mathbb{D}.$ (Here, we assume k and the radial projection of S:

$$PrS = \left\{\frac{z}{|z|} : z \in S\right\}.$$

Then we have

$$J(Z;\lambda_\alpha) \le \inf_{S\subset Z}\{\alpha[\hat{\kappa}(PrS) + \log\hat{\kappa}(PrS)] - T(s) + \alpha\log^+ T(s)\} + C_\alpha$$

and

$$J(Z;\lambda_\alpha) \ge \inf_{S\subset Z}\{\alpha[\hat{\kappa}(PrS) - \log\hat{\kappa}(PrS)] - T(s)\} - C_\alpha$$

where $C_\alpha > 0$ depends only on α; and the infima are taken over all finite subsets S of Z.

<u>COROLLARY</u> 3.1 *For a sequence Z such that 0 is not in Z; define*

$$D^+(Z) = \inf\{m : \inf_{S\subset Z}(m\hat{\kappa}(PrS) - T(s)) > -\infty\}.$$

Then $D^+(Z) \le \alpha$ is necessary and $D^+(Z) < \alpha$ is sufficient for Z to be an $A^{-\alpha}$ zero set.

Note that for other spaces $A^{<k>}$ such that k has faster than logarithmic growth, a similar description of zero sets is not known.

4 <u>Problem (B) for $A^{<k>}$</u>

<u>THEOREM</u> 4.1

$$\begin{aligned} J(E;\varphi;k) \le & \int_E \max\{\log\varphi(\zeta), \log p\}dm(\zeta) - (\log p)\frac{\alpha}{1-\alpha}(1-|E|) \\ & + \left(\frac{L}{\alpha}\right)^{\log_2 C} Entr_k(E) \end{aligned}$$

where $0 < p \le 1$, $0 < \alpha \le \frac{1}{2}$ are arbitrary, C is the constant in (3), L is an absolute constant, and $Entr_k(E)$ is the k-entropy of E, defined as follows:

$$Entr_k(E) = \sum_n \int_{1-|I_n|}^{1} k(t)dt$$

where $\{I_n\}$ are the complementary arcs of E.

Special cases: (1) $E = \partial\mathbb{D}$. Letting $p \to 0^+$, we get

$$J(\partial\mathbb{D}, \varphi, k) \le \int_{\partial\mathbb{D}} \log \varphi(\zeta)\, dm(\zeta)$$

which is the classical Jensen's inequality (in fact, equality.)
(2) If $0 \le \varphi(\zeta) \le 1$ on E and $p = \max_{\zeta \in E} \varphi(\zeta)$, we obtain

$$J(E, \varphi, k) \le (\log p)\frac{|E| - \alpha}{1 - \alpha} + \left(\frac{L}{\alpha}\right)^{\log_2 C} Entr_k(E).$$

Choosing $\alpha = |E|/2$, we get

$$J(E, \varphi, k) \le \frac{1}{2}(\log p)|E| + \left(\frac{2L}{|E|}\right)^{\log_2 C} Entr_k(E).$$

(3) If $p = 1$ and $\alpha = \frac{1}{2}$, then

$$J(E, \varphi, k) \le \int_E \log^+ \varphi(\zeta)\, dm(\zeta) + (2L)^{\log_2 C} Entr_k(E).$$

Proof: Write

$$\partial\mathbb{D} - E = \bigcup_n I_n$$

where the I_n are open disjoint arcs on the unit circle. Call a_n and b_n the endpoints of I_n. Let $0 < \alpha \le \frac{1}{2}$. Let γ_n be the open arc of the circle inside the unit disk passing through a_n and b_n and forming an angle of $\pi\alpha$ (we will think of it as the normalized angle α) with the arc I_n. Let $\Gamma = \bigcup_n \gamma_n$. $\Gamma \cup E$ forms the boundary of an open subset Ω of the unit disk containing the origin. For the proof, we construct three functions U_1, U_2, and V as follows.
Step 1: Construction of U_1 and U_2.
Define

$$U_1(z) = \int_E Re\left(\frac{\zeta + z}{\zeta - z}\right) dm(\zeta).$$

U_1 is the harmonic measure of E with respect to $\mathbb{D}$.

<u>LEMMA</u> 4.1

$$\lim_{r \to 1^-} U_1(r\zeta) = \chi_E(\zeta) \ a.e.\ on\ \partial\mathbb{D}$$

where χ_E is the characteristic function of E. In addition, $U_1(z) \le \alpha$ for $z \in \Gamma$.

Assume $Entr_k(E)$ is ﬁnite and deﬁne the following harmonic function

$$V(z) = \int_{\partial \mathbf{D}}$$

By relabeling $L;$ we get the statement of the lemma. $\Box$

Step 3: Construction of H and application of the maximum principle. Finally, let us define

$$H(z) = U_2(z) - (\log p)\frac{\alpha}{1-\alpha}(1 - U_1(z)) + (^{L}$$

$\lim_{r \to 1^-} f(r\zeta$